Comment on "Stability and quality factor of a one-dimensional subwavelength cavity resonator containing a left-handed material"

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In Ref. 1 (later on referred to as Engheta's paper), Engheta studied the dispersion relation of a one-dimensional (1D) cavity resonator with electrically perfectly conducting walls (mirrors) and in between two slabs of dielectrics whose permittivity, permeability, and refractive index are ϵ_1, μ_1, and n_1 and ϵ_2, μ_2, and n_2, respectively, as we see in Fig. 1. The dispersion relation is, as derived in Engheta's paper and cited in Ref. 2 (later on referred to as Shen's paper),

$$\frac{\tan(n_1kd_1)}{\tan(n_2kd_2)} = -\frac{n_1\mu_2}{n_2\mu_1}, \tag{1}$$

where k is the wave number in free space. When both of the slabs are made of conventional dielectrics, the dispersion relation gives no solution when the size of the cavity is much smaller than the operating wavelength, as has been known for many years. The major contribution in Engheta's paper is to point out that when one of the slabs is made of a metamaterial with negative permittivity and negative permeability (thus a sufficient condition for a negative refractive index), the dispersion relation Eq. (1) has a solution for a cavity of deep subwavelength size. Furthermore, Engheta shows that when the size of the cavity is very small, the solution should satisfy

$$\frac{d_1}{d_2} \cong -\frac{\mu_2}{\mu_1}. \tag{2}$$

The effectiveness and precision of the former equation were questioned in Shen's paper. After some relatively complicated mathematical transforms, the authors concluded that d_1/d_2 would be strictly equal to $-\mu_2/\mu_1$ only when the impedances of the two layers were matched. While for a general case when the impedances are not matched, d_1/d_2 deviates obviously from $-\mu_2/\mu_1$, as shown in Fig. 2(a) of Shen's paper.

Although we basically agree with the numerical results shown in Fig. 2(a) of Shen's paper, we have very different interpretations of it. In fact, this figure gives the deviation of d_1/d_2 from $-\mu_2/\mu_1$ (which is $\bar{\varsigma}-1$ in the figure) of cavity compactness (N) for different impedance-matching situation (different $\hat{\epsilon}=-\epsilon_2/\epsilon_1$, since $\hat{\mu}=-\mu_2/\mu_1$ is assumed). It can be noticed that even for the worst case shown in the figure ($\hat{\epsilon}=4, N=10$), we still have $d_1/d_2/(-\mu_2/\mu_1)=1.1$, not severely deviated from 1. It is true that the deviation of $d_1/d_2/(-\mu_2/\mu_1)$ from 1 has "orders of magnitude change" when the impedance mismatch changes, as pointed out in their text and emphasized by the logarithmic plot of Fig. 2(a) in Shen's paper. However, they are all small values with different orders. After all, Eq. (2) is satisfied in an approximate sense and is a deduction from the strict dispersion relation Eq. (1). The purpose of Eq. (2) is to show the obvious difference between a conventional cavity and the one with metamaterial, rather than to be used for resonating frequency calculation.

To argue whether 0.1 is a small value or not gives no physical significance, thus in the text below we give an analytical formula connecting d_2/d_1 with μ_1/μ_2 in a manner more precise than that of Eq. (2). Considering a given metamaterial design and at a given operating frequency, we try to design the geometry of the cavity so that it resonates. Mathematically this is to fix n_1, μ_1, n_2, μ_2, and k, and search for d_1 and d_2 that satisfy Eq. (1). From Eq. (1) we can have

$$d_2 = \frac{1}{n_2k}\{\tan^{-1}[\alpha \tan(n_1kd_1)] + m\pi\}, \tag{3}$$

where $\alpha=-n_2\mu_1/n_1\mu_2$ and m is an integer. Notice that in order to achieve Eq. (3) we make no approximation, and it is valid either when both the two slabs are conventional dielectrics or when one of them is a metamaterial. Equation (3) connects the thickness of the second slab to the thickness of the first slab in order for the structure to resonate at the given frequency. At this point, it is easy to see that when both slabs are conventional dielectrics (i.e., $n_{1,2}>0$ and $\mu_{1,2}>0$, $\alpha<0$), for small enough d_1, $\tan^{-1}(\alpha \tan(n_1kd_1))<0$, the smallest possible d_2 is achieved for $m=1$, implying that the total size of the cavity is comparable to the wavelength. However, when $n_1>0$, $\mu_1>0$ and $n_2<0$, $\mu_2<0$, we still have $\alpha<0$. Under this condition, for arbitrarily small d_1, $\tan^{-1}(\alpha \tan(n_1kd_1))/(n_2k)>0$. Thus the smallest solution of d_2 for Eq. (3) is achieved for $m=0$, which gives a resonating cavity of subwavelength total size. This is the major conclusion given in Engheta's paper.

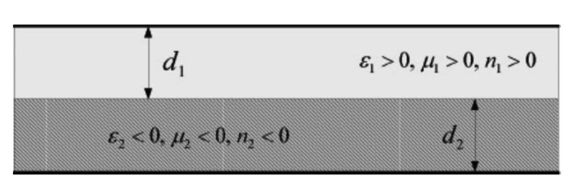

FIG. 1. The 1D subwavelength cavity considered in Refs. 1 and 2.

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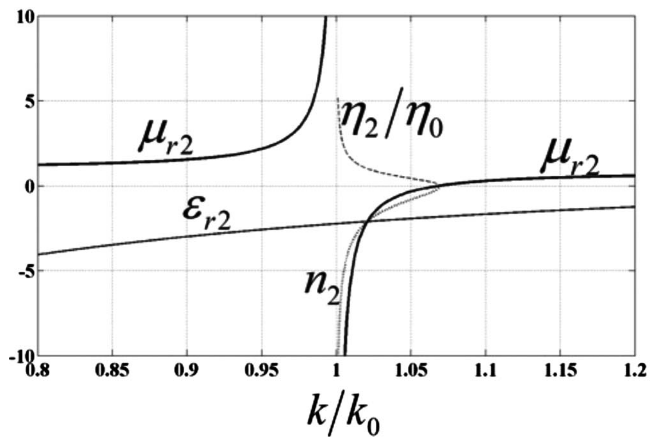

FIG. 2. ϵ_{r2}, μ_{r2}, and n_2 and η_2/η_0 vs frequency.

When kd_1 is small, the $\tan^{-1}(\cdot)$ function can be expanded into Taylor series, thus Eq. (3) gives (notice $m=0$)

$$d_2 = \frac{n_1}{n_2}\alpha d_1 + \frac{\alpha-\alpha^3}{3}\frac{n_1^3}{n_2 k}(kd_1)^3 + O(kd_1)^5, \tag{4}$$

so we can get

$$\frac{d_2}{d_1} = -\frac{\mu_1}{\mu_2} + \frac{\alpha-\alpha^3}{3}\frac{n_1^3}{n_2}(kd_1)^2 + O(kd_1)^4. \tag{5}$$

The equation above is an extension of Eq. (2) with higher precision. It is obvious now that when kd_1 is small, d_2/d_1 is different from $-\mu_1/\mu_2$ by a small quantity of order $(kd_1)^2$, while other parameters (the impedances, the relative permittivity, and permeability, etc.,) only influence the constants in front of $(kd_1)^2$. Notice that the equation above gives basically the same information as Eq. (6) in Shen's paper, but we believe in a more straightforward way. The discussion in Shen's paper indeed correctly revealed the difference between d_1/d_2 and $-\mu_2/\mu_1$ in the sense of the exact quantity. However, we would like to point out that such difference is the terms of $(kd_1)^2$ and higher orders in Eq. (5), and will vanish quickly as the size of the cavity decreases further. Based on the analysis above, we can see that Eq. (2) is indeed valid in the approximate sense, and reveals an important property of such a cavity that deserves great attention.

In the next part of Shen's paper the stability of such a subwavelength cavity is discussed, and the cavity is called unstable if the derivatives of the relative resonant frequency shift with respect to some design parameters shows any singular behavior. Such a definition of stability is, however, not compatible with the definition in some existing references e.g., Ref. 3 where the stability is tied to the quality factor of the resonator. A more appropriate term for the resonant frequency variation with cavity parameters is "resonant frequency tolerance." However, we will still use the term "stability" in the rest of the discussion in the sense as that in Shen's paper. It is found in Shen's paper that the cavity cannot be stable when the impedances of the two slabs are matched, which we do not agree to. In the following, we study the stability of this structure with much simpler mathematics. We show that the necessary frequency dispersion of the metamaterial cannot be neglected in this discussion, and we show that there is no intrinsic instability issue for this structure, whether the impedances are matched or not.

The design parameters of such a cavity include d_1, d_2, n_1, and μ_1, and the properties of the metamaterial which are determined by its own design parameters. Since these design parameters are independent of each other, we can study the change in the resonant frequency as a function of the perturbation of each parameter respectively, with all the others fixed as constants at the time. Suppose we first study the change in the resonant frequency vs change in d_2. From Eq. (1), we have $\tan(n_1kd_1) = -\eta_2\tan(n_2kd_2)/\eta_1$ where $\eta_{1,2} = \sqrt{\mu_{1,2}/\epsilon_{1,2}} > 0$. If we take the full derivative of this equation, after some simple mathematics we have,

$$\left(\frac{n_1d_1}{\cos^2(n_1kd_1)} + \frac{\eta_2}{\eta_1}\frac{n_2d_2}{\cos^2(n_2kd_2)}\right)\Delta k = -\frac{\eta_2}{\eta_1}\frac{n_2k}{\cos^2(n_2kd_2)}\Delta d_2. \tag{6}$$

Notice when deriving Eq. (6) the temporal dispersion of the metamaterial is neglected. This is also the assumption in Shen's paper when discussing the stability issue. This assumption may be effective, since for a small perturbation of the design parameter the change in the resonant frequency may also be small. However, such an assumption should always be carefully examined, as we will show later.

From Eq. (6), we notice that the stability of the cavity resonator is in general good except for the impedance-matched case, for which we have $n_1d_1 = -n_2d_2$ and $\eta_1 = \eta_2$. For this case, the coefficient of Δk is zero, thus Δk diverges for any perturbation in d_2. We see that for the impedance-matched case our Eq. (6) gives the same conclusion as their analysis when we use the same assumption (dispersionless metamaterial), but with much simpler mathematics. In fact, there is an intuitive explanation to this phenomenon. When the impedance of the two slabs inside the cavity is matched, the resonant condition simplifies so that the round trip of the wave inside the cavity gives $2m\pi$ phase delay, or $k(n_1d_1 + n_2d_2) = m\pi$. When both slabs are conventional dielectrics, for the smallest resonating cavity we have $m=1$, and when d_2 deviates above/below the designed value, the phase delay condition can still be satisfied if only the operating wavelength decreases/increases to a comparable amount, which means the cavity is stable. However, if one of the slab is a disperseless metamaterial with $n_2 < 0$, for the smallest resonating cavity we have $m=0$, thus for any small change in d_2 from the designed value, the round trip phase delay is $2k\Delta d_2$ (notice we are assuming that the metamaterial is disperseless), and the phase condition of the resonator cannot be satisfied unless $2k\Delta d_2$ equals nonzero integral times of π, at which the cavity is resonating at a high-order mode. In other words, any small change in d_2 destroys the lowest-resonating mode for which the cavity size is small compared to the wavelength, if we assume the metamaterial is dispersionless.

However, the dispersionless assumption for metamaterials should always be examined carefully especially when some extreme phenomenon happens. For the impedance-matched case in Eq. (6), k changes a lot for a small perturbation of d_2, indicating that the material dispersion should be taken into account. Suppose slab 1 is a conventional dielectric while slab 2 is a metamaterial, and we again take the full derivatives of $\tan(n_1kd_1)=-\eta_2\tan(n_2kd_2)/\eta_1$, but this time we retain the terms of Δn_2 and $\Delta\eta_2$, and relate them to Δk through the dispersion property of ϵ_2 and μ_2, then we have

$$\left[\frac{n_1d_1}{\cos^2(n_1kd_1)}+\frac{\eta_2}{\eta_1}\frac{n_2d_2}{\cos^2(n_2kd_2)}+\frac{\eta_2}{2\eta_1}\frac{n_2kd_2}{\cos^2(n_2kd_2)}\left(\frac{\mu_2'}{\mu_2}+\frac{\epsilon_2'}{\epsilon_2}\right)+\frac{\eta_2}{2\eta_1}\tan(n_2kd_2)\left(\frac{\mu_2'}{\mu_2}-\frac{\epsilon_2'}{\epsilon_2}\right)\right]\Delta k=-\frac{\eta_2}{\eta_1}\frac{n_2k}{\cos^2(n_2kd_2)}\Delta d_2, \tag{7}$$

where ϵ_2' and μ_2' are the derivatives of ϵ_2 and μ_2 with respect to k, and when they are set to zero, Eq. (7) is the same as Eq. (6). The expressions for the stability due to the perturbation of other parameters are similar to that of Eq. (7). In fact, the coefficient of Δk is the same, but only the right-hand side (RHS) is different for different parameters. Thus the singularity (if there are any) property of Δk will be similar for different design parameters. From Eq. (7) we can see that the change in the resonance frequency for any given perturbations in d_2 (or other parameters which can be discussed similarly) is indeed influenced by the frequency property of ϵ_2 and μ_2, and there is no intrinsic singular point for Δk. Specifically, the singularity in Eq. (6) for the impedance-matched case disappears in Eq. (7) due to the material dispersion.

We give some numerical examples in the following. Considering a cavity with the first slab, the free space ($\epsilon_1=\epsilon_0$, $\mu_1=\mu_0$, $n_1=1$, and $\eta_1=\eta_0$), and the second slab, a metamaterial with the effective constitutive parameters can described by Drude (or Drude-Lorentz) model as

$$\epsilon_2=\epsilon_0\left(1-\frac{k_p^2}{k^2}\right),$$

$$\mu_2=\mu_0\left(1-\frac{Fk^2}{k^2-k_0^2}\right), \tag{8}$$

with $F=0.1257$ and $k_p=1.8k_0$. In reality, such a metamaterial is realized in radio frequency domain by combining the metallic wire medium and the split ring resonators.[4–6] The relative permittivity and relative permeability as a function of frequency is shown in Fig. 2, together with the impedance (normalized to the free space impedance η_0) and refractive index in the frequency band when both the permittivity and the permeability are negative. We can design the thickness of the two slabs so that the cavity resonates at a given operating frequency, and here we consider two cases. For one, the operating frequency is $k=k_1=1.021k_0$ when $\epsilon_{r2}=\mu_{r2}=-2.109$ and $\eta_2=\eta_1=\eta_0$ (the impedance-matched case), and for the other $k=k_2=1.033k_0$ when $\mu_{r2}=-1$, $\eta_2=0.701\eta_0$. For any given d_1, we determine d_2 from Eq. (3) with $m=0$, and the results for the two cases are shown in Fig. 3 as solid lines. Notice that here we plot the size relative to the operating wavelength ($\lambda_{1,2}$ for k_1 or k_2) is plot. The dashed lines in the figure are $-\mu_1d_1/\mu_2$, and are for comparison purpose. From this we can see that the relation of d_1 and d_2 converges to Eq. (2) when the size of the cavity is much smaller than the operating wavelength. Specifically, for the impedance-matched case, d_2/d_1 is strictly equal to $-\mu_1/\mu_2$ for any given d_1. It is interesting to point out that, since $|n_2|$ is relatively small ($n_2=-2.109$, -1.427 for the two cases, respectively), Fig. 3 also indicates that the cavity is "electrically small," i.e., $n_1d_1+|n_2|d_2\ll\lambda_{1,2}$. This is not possible when conventional dielectrics are used, even with materials of large refractive index.

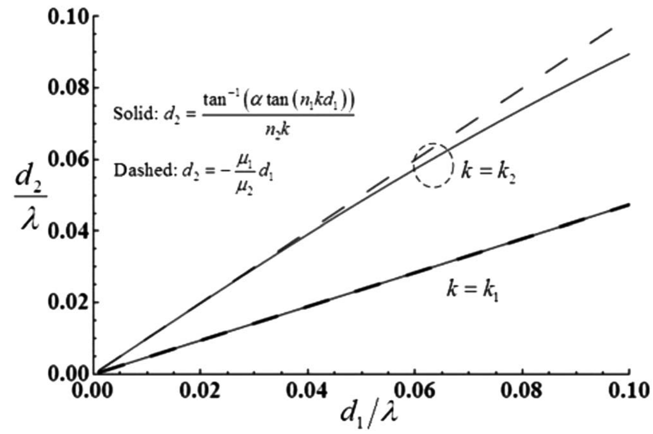

FIG. 3. Value of d_2 for given d_1 when resonating at a certain operating frequency, normalized to the operating wavelength, respectively. Bold: $k=k_1$, Light: $k=k_2$.

In the former paragraph we designed the size of the cavity for given operating frequencies. It is interesting to see the real "frequency dispersion" of such type of cavities. To do this, we rewrite the dispersion equation Eq. (1) as

$$\eta_1\tan(n_1kd_1)=-\eta_2\tan(n_2kd_2) \tag{9}$$

and we plot the value of the left- and right-hand side as k varies, for given cavity geometry designs (d_1 and d_2), as we see in Fig. 4. Here the sizes of d_1 and d_2 are given in $\lambda_0=2\pi/k_0$, where k_0 is a constant used in the metamaterial design [(Eq. (8)]. Obviously, the two crosses actually indicate two cavity designs that resonate at different frequencies,

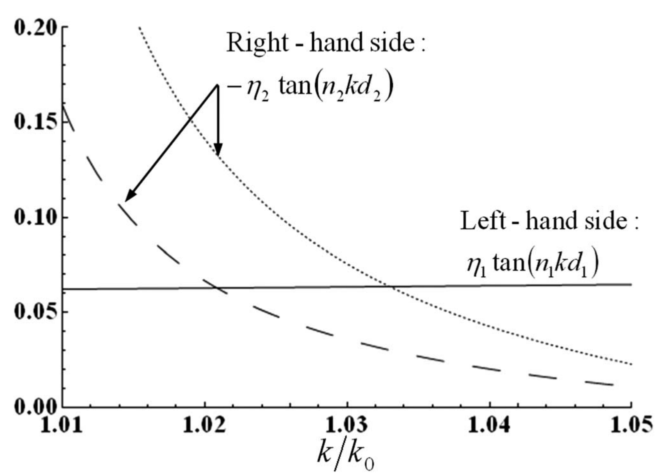

FIG. 4. Solid: the left-hand side of Eq. (9), $d_1=9.80\times10^{-3}\lambda_0$; dashed: the RHS of Eq. (9), for $d_2=4.64\times10^{-3}\lambda_0$; dotted: the right-hand side of Eq. (9), for $d_2=9.78\times10^{-3}\lambda_0$.

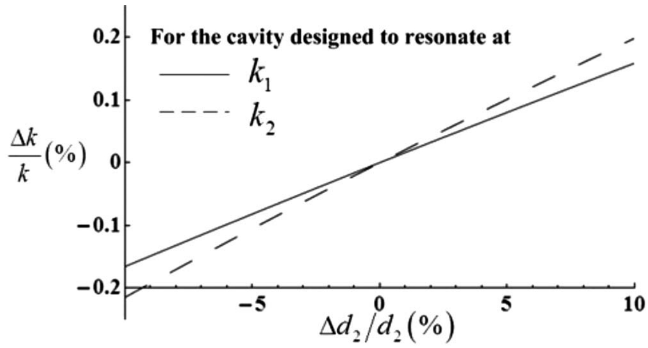

FIG. 5. $\Delta k/k$ vs $\Delta d_2/d_2$, indicating the stability of the cavity.

both of which are electrically small, which can be easily tested.

The sensitivity of the operating frequency to the variation in d_2 can be calculated from Eq. (7) when we have the frequency parameters of the metamaterial. Consider the specific cases when $d_1=0.1\lambda_{1,2}$ for the two design cases. From Eq. (7) we have $\Delta k/k=0.016\Delta d_2/d_2$ for the cavity resonating at k_1, and $\Delta k/k=0.021\Delta d_2/d_2$ for the cavity resonating at k_2. The relative change in the operating frequency is even much smaller than that of d_2, due to the obvious dispersion of the metamaterial. The value of $\Delta k/k$ for different d_2 deviations is shown in Fig. 5. To achieve this figure, the resonating frequency for any d_2 deviating from the desired value is calculated from Eq. (1) numerically with the dispersion property of n_2 and μ_2 taken into account, from which we get $\Delta k/k$. We can see that the cavities are indeed quite stable for the design we proposed here.

The rest of Shen's paper discuss the quality factor of such a cavity when material loss in the metamaterial exists. We have no comments for this part.

In the former discussion the only assumptions are that the metamaterial is lossless and spatially uniform, which are the same assumptions used in Engheta's paper and in Shen's paper. It is true that the unit cells of the metamaterials designed and fabricated so far are still not small enough to allow the metamaterials to be treated rigorously as homogeneous materials. However, cavity miniaturization using discrete metamaterial inclusions prompted by this idea were demonstrated,[7] which justify the value of this idea. In conclusion, the solution of the dispersion relation of a 1D cavity considered by Engheta in Ref. 1 indeed approaches Eq. (2) as the size decreases. The difference is of second-order small value $(kd_1)^2$. The temporal dispersion of the metamaterial cannot be neglected when discussing the stability of the solution and they are in general stable. The general results in Engheta's paper, confirmed by this comment, could have significant practical implications for making subwavelength resonators.

During the preparation of this manuscript we exchanged opinions constructively with the authors of the paper commented here. We appreciate very much their open-minded discussions.

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